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UNIQUENESS OF MINIMAL GENUS SEIFERT SURFACES FOR LINKS

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We study the method of deciding whether the minimal genus Seifert surfaces for the given link in the 3-sphere are unique. We give a sufficient condition for the uniqueness by using product decompositions and A-operations of sutured manifolds. We give a necessary and sufficient condition in case when the link is the boundary of a Murasugi sum of two minimal genus Seifert surfaces.

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links

1. Introduction

Let L be an oriented link in the 3-sphere S^3, $E(L)=S^3-\operatorname{Int} N(L)$ the exterior of L, and S a Seifert surface for L, i.e., $\partial S=L$. Then we often abbreviate $E(L)\cap S$ by denoting S. Two Seifert surfaces S_1 and S_2 for L are *equivalent* if S_1 is ambient isotopic to S_2 in $E(L)$. We say that an oriented Seifert surface S for L is of *minimal genus* if no component of S is closed, and $\chi(R)\leq\chi(S)$ for any Seifert surface R for L satisfying the above condition. Then several authors showed that there exist knots whose minimal genus Seifert surfaces are not unique under the above equivalence [1, 2, 8, 12]. On the other hand, it is known that there are nonfibered knots whose minimal genus Seifert surfaces are unique [13, 19].

In this paper we study the method of deciding whether the minimal genus Seifert surfaces of the given links are unique. Our arguments heavily depend on the sutured manifold of Gabai [4]. We note that the incompressible Seifert surfaces for fibered links are unique, and, in [6], Gabai gave a criterion for deciding whether the given link is fibered, by using product decompositions of sutured manifolds. In Section 4, we will give a sufficient condition of the minimal genus Seifert surfaces for the given links being unique, by using product decompositions and A-operations (see

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Section 2) of sutured manifolds Especially we give a necessary and sufficient condition in case when the given link has a minimal genus Seifert surface S such that $S^3 - \mathrm{Int}\, N(S)$ is a handlebody (Theorem 4.6) In [3], Gabai proved that the Murasugi sum of minimal genus surfaces (fibered links respectively) is minimal genus (fibered respectively) In Section 5, we show

Theorem 5.1. *Let L_i $(i = 1, 2)$ be a link with a minimal genus Seifert surface R_i, R a Murasugi sum of R_1 and R_2. Then the minimal genus Seifert surfaces for $L = \partial R$ are unique if and only if one of L_1, L_2, say L_2, is fibered and the minimal genus Seifert surfaces for L_1 are unique*

We note that Theorem 5.1 implies a result of Boileau-Gabai [6, Corollary 3.2] As a consequence of Theorem 5 1, we have.

Corollary 5.2 (cf [7, 1 18, 1 19]). *Let L be an arborescent link [7] with the total weight at each vertex is even and nonzero. Then the minimal genus Seifert surfaces for L are unique if and only if the total weights of all but one vertices are ± 2*

We note that 2-bridge knots are arborescent links with trees such that the total weight at each vertex is even and nonzero, and each vertex has at most two adjacent edges [8, Fig 2] Hence Corollary 5 2 is a natural generalization of a result of Hatcher and Thurston [8, corollary]

By using the results of Section 4 together with Theorem 5 1, we could give the list of knots of ≤ 10 crossings whose minimal genus Seifert surfaces are not unique (Appendix) Especially the minimal genus Seifert surfaces for 9_{25} are unique The fact will be used in [11] to show that the unknotting number of 9_{25} is 2.

2. Preliminaries

Throughout this paper, we work in the piecewise linear category and manifolds are oriented unless otherwise specified For the definitions of standard terms of 3-dimensional topology, knot and link theory, see [9, 14].

Firstly, we recall the definition of the sutured manifold [4] A *sutured manifold* (M, γ) is a compact 3-dimensional manifold M together with a set γ $(\subset \partial M)$ of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$ In this paper, we mainly treat the case of $T(\gamma) = \emptyset$ The core curves of $A(\gamma)$, $s(\gamma)$, are the *sutures* Every component of $R(\gamma) = \partial M - \mathrm{Int}\, \gamma$ is oriented, and $R_+(\gamma)$ ($R_-(\gamma)$ respectively) denotes the union of the components whose normal vectors point out of (into respectively) M Moreover the orientations of $R(\gamma)$ must be coherent with respect to $s(\gamma)$

Let (M, γ) be a sutured manifold. A properly embedded disk D $(\subset M)$ is a *product disk* if ∂D intersects $s(\gamma)$ transversely in two points. A *product decomposition* is a sutured manifold decomposition [4, Definition 3.1], $(M, \gamma) \overset{D}{\rightsquigarrow} (M', \gamma')$, where D is a product disk (see Fig. 1)

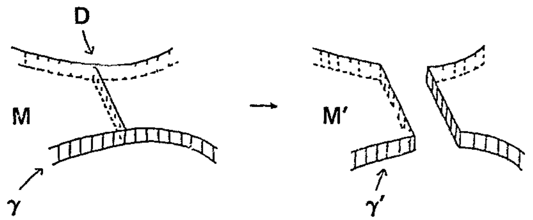


Fig 1

We say that the sutured manifold (M, γ) is a *product sutured manifold* if (M, γ) is homeomorphic to $(F\times I, \partial F\times I)$, where F is a 2-manifold and $R_+(\gamma)=F\times\{1\}$, $R_-(\gamma)=F\times\{0\}$. We say that (M, γ) is an *almost product sutured manifold* if every incompressible surface F in M with $\partial F=s(\gamma)$ is parallel to $R_+(\gamma)$ or $R_-(\gamma)$. We note that possibly there is an incompressible surface F in M with $\partial F=s(\gamma)$ such that some components of F are parallel to some components of $R_+(\gamma)$ and the rest of the components are parallel to some components of $R_-(\gamma)$ even if (M, γ) is an almost product sutured manifold. It is known that a product sutured manifold is an almost product sutured manifold In fact, if F is an incompressible surface in a product sutured manifold (M, γ) with $\partial F=s(\gamma)$, then F is parallel to $R_+(\gamma)$ and $R_-(\gamma)$

Let (M_1, γ_1) be a sutured manifold with M_1 irreducible, $(M_1, \gamma_1)\overset{D}{\rightarrow}(M_2, \gamma_2)$ a product decomposition Then we have

Lemma 2.1. *Suppose that there is an incompressible surface S_1 in M_1 with $\partial S_1=s(\gamma_1)$ and S_1 not parallel to $R_+(\gamma_1)$ and $R_-(\gamma_1)$, i.e., (M_1, γ_1) is not an almost product sutured manifold Then there is an incompressible surface S_2 in M_2 with $\partial S_2=s(\gamma_2)$, S_2 not parallel to $R_+(\gamma_2)$ and $R_-(\gamma_2)$, and $\chi(S_2)=\chi(S_1)+1$*

Proof. Since M_1 is irreducible and S_1 is incompressible, by using standard cut and paste arguments, we may suppose that $D\cap S_1$ consists of an arc a. By cutting S_1 along a, we get a surface S_2 in M_2 such that $\partial S_2=s(\gamma_2)$. Since S_1 is incompressible, S_2 is incompressible Since S_1 is not parallel to $R_+(\gamma_1)$ and $R_-(\gamma_1)$, S_2 is not parallel to $R_+(\gamma_2)$ and $R_-(\gamma_2)$ And, clearly, we have $\chi(S_2)=\chi(S_1)+1$. □

Lemma 2.2. *Suppose that there is an incompressible surface S_2 in M_2 with $\partial S_2=s(\gamma_2)$, and S_2 not parallel to $R_+(\gamma_2)$ and $R_-(\gamma_2)$. Then there is an incompressible surface S_1 in M_1 with $\partial S_1=s(\gamma_1)$, S_1 not parallel to $R_+(\gamma_1)$ and $R_-(\gamma_1)$, and $\chi(S_1)=\chi(S_2)-1$.*

Proof. By tracing the construction of S_2 from S_1 in the proof of Lemma 2 1 conversely, we can get an incompressible surface S_1 in M_1 with $\partial S_1=s(\gamma_1)$, which is not parallel to $R_+(\gamma_1)$ and $R_-(\gamma_1)$ Clearly $\chi(S_1)=\chi(S_2)-1$. □

Let (M, γ) be a sutured manifold with M irreducible, a $(\subset\partial M)$ an arc such that $a\cap R(\gamma)$ an arc Then either $a\cap R_+(\gamma)=\emptyset$ or $a\cap R_-(\gamma)=\emptyset$. We suppose that

$a \cap R_-(\gamma) = \emptyset$. The other case is treated the same way. Let a' be an arc properly embedded in $\partial M - \operatorname{Int} R_-(\gamma)$ such that $a' \cap R_+(\gamma) = a \cap R_+(\gamma)$. Then $\partial N(a') \cap R_+(\gamma)$ consists of two arcs a_1, a_2, which are parallel to $a \cap R_+(\gamma)$ in $R_+(\gamma)$, where $N(.)$ denotes a regular neighborhood in ∂M Let $\gamma' = (\gamma - N(a')) \cup N(a_1 \cup a_2)$. Then γ' is a new suture on M and $s(\gamma')$ has a natural orientation induced from $s(\gamma)$ (see Fig 2) We say that (M, γ') is obtained from (M, γ) by an *A-operation* along a

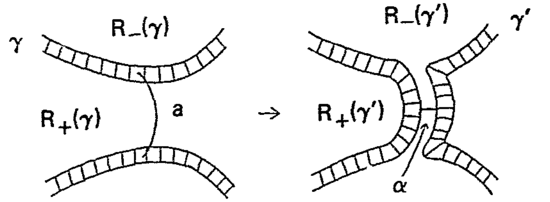


Fig 2

Let S be a surface properly embedded in M with $\partial S = s(\gamma)$. Suppose that there is a disk Δ ($\subset M$) with the following properties:

(i) $\Delta \cap \partial M = a$ an arc, $\Delta \cap S = b$ an arc such that $\partial a = \partial b$, $a \cup b = \partial \Delta$,

(ii) $a \cap R(\gamma)$ is an arc

Then we can apply an A-operation on (M, γ) along a to get (M, γ'), and we get a surface S', by doing surgery on S along Δ, such that $\partial S' = s(\gamma')$. Then we have·

Lemma 2.3. *If S' is parallel to $R_\varepsilon(\gamma')$ ($\varepsilon = +$ or $-$), then either S is parallel to $R_\varepsilon(\gamma)$ or S is compressible in M.*

Proof. Let V be the closure of the component of $M - S'$ such that $(V, S', R_\varepsilon(\gamma'))$ is homeomorphic to $(S' \times I, S' \times \{0\}, S' \times \{1\})$ Let α be an arc properly embedded in the closure of the component of $\partial M - s(\gamma')$ intersecting a transversely in one point, which corresponds to a dual arc of a (see Fig. 2). Let $N(\alpha)$ be a regular neighborhood of α in the closure of the component of $\partial M - s(\gamma')$, S'' the surface obtained by moving $S' \cup N(\alpha)$ slightly so that $S'' \cap \partial M = \partial S''$ Then S is ambient isotopic to S''. If $V \cap \operatorname{Int} \alpha = \emptyset$, then S'' is parallel to $R_\varepsilon(\gamma)$ Suppose that $\alpha \subset V$, i e., $\alpha \subset R_\varepsilon(\gamma')$. Let l be the simple loop on S'' which corresponds to $(\alpha \times \{0\}) \cup (\partial\alpha \times I) \cup (\alpha \times \{1\})$ on $S' \cup N(\alpha)$. Then l is an essential simple loop on S and l bounds a disk corresponding to $\alpha \times I$. Hence S is a compressible surface. $\square$

Lemma 2.4. *Suppose that S' is not parallel to $R_+(\gamma')$ and $R_-(\gamma')$, and S is parallel to $R_\varepsilon(\gamma)$ ($\varepsilon = +$ or $-$). Then $R_\varepsilon(\gamma')$ is compressible in M and there is a product disk D for (M, γ) such that $\Delta \subset D$*

Proof. Let V be the closure of the component of $M-S$ such that (V, S) is homeomorphic to $(S\times I, S\times\{0\})$, i.e., $R_\varepsilon(\gamma)\subset\partial V$. Since S' is not parallel to $R_+(\gamma')$ and $R_-(\gamma')$, Δ is not contained in V. Let Δ' $(\subset V)$ be the disk corresponding to $(\partial\Delta\cap S)\times I$, and $D=\Delta\cup\Delta'$. Then ∂D is an essential loop of $R_\varepsilon(\gamma')$ Hence $R_\varepsilon(\gamma')$ is compressible Moreover D is a product disk for (M, γ). □

Let D be an essential disk properly embedded in M with $\partial D\cap s(\gamma)\neq\emptyset$. Then $\partial D-s(\gamma)$ consists of $2n$ arcs. If necessary, by moving D by an ambient isotopy, we may suppose that $\partial D-\gamma$ also consists of $2n$ arcs. Then we have:

Lemma 2.5. *Suppose that there is an incompressible surface S in M with $\partial S=s(\gamma)$, which is not parallel to $R_+(\gamma)$ and $R_-(\gamma)$. Then there exist at least two components, $a_1,\ldots,a_m$, of $\partial D-s(\gamma)$ such that:*

(i) $\mathrm{cl}\, a_i\cap\mathrm{cl}\, a_j=\emptyset$ $(i\neq j)$.

(ii) *Suppose that (M, γ_i) is obtained from (M, γ) by an A-operation along a_i. Then there is an incompressible surface S_i $(\subset M)$ with $\partial S_i=s(\gamma_i)$, S_i not parallel to $R_+(\gamma_i)$ and $R_-(\gamma_i)$, and $\chi(S_i)=\chi(S)+1$.*

Proof. Since M is irreducible, by using cut and paste arguments, we may suppose that $D\cap S$ consists of transverse arcs. Then there exist at least two innermost disks, $\Delta_1,\ldots,\Delta_m$, in D. Let $a_i=\mathrm{Int}(\Delta_i\cap\partial M)$. Then, clearly, we have $\mathrm{cl}\, a_i\cap\mathrm{cl}\, a_j=\emptyset$ $(i\neq j)$. Let S_i be the surface obtained from S by doing a surgery along Δ_i. Then, by Lemma 2 3, we see that S_i satisfies the conclusions of Lemma 2.5. □

Proposition 2.6. *Let (M, γ) be a sutured manifold, and $(M^1, \gamma^1),\ldots,(M^n, \gamma^n)$ sutured manifolds obtained from (M, γ) by applying a sequence of product decompositions and all possible A-operations along subarcs of the boundary of disks in the ambient manifolds If every (M^i, γ^i) is an almost product sutured manifold, then (M, γ) is an almost product sutured manifold*

Remark. By Lemma 2.5(i), we need not consider all possible A-operations. See Example 2 9 below.

Proof of Proposition 2.6. Assume that (M, γ) is not an almost product sutured manifold, i.e., there is an incompressible surface S properly embedded in M such that $\partial S=s(\gamma)$, and S is not parallel to $R_+(\gamma)$ and $R_-(\gamma)$. Then, by Lemmas 2.1, 2 3. 2.5, we have an incompressible surface S' properly embedded in some M^i such that $\partial S'=s(\gamma^i)$, and S' not parallel to $R_+(\gamma^i)$ and $R_-(\gamma^i)$, a contradiction. □

Example 2.7. Let (M, γ) be a sutured manifold such that M is irreducible, γ is an annulus with $s(\gamma)$ contractible in ∂M. Then (M, γ) is an almost product sutured manifold.

Proof. We note that $R(\gamma)$ consists of a disk and ∂M with one hole. Let S ($\subset M$) be an incompressible surface with $\partial S = s(\gamma)$. Since $s(\gamma)$ is contractible in ∂M, S is a disk. Since M is irreducible, we see that S is parallel to the disk component of $R(\gamma)$ □

Example 2.8. Let (M, γ) be a sutured manifold such that M is a solid torus $D^2 \times S^1$, γ consists of two annuli with each component of $s(\gamma)$ not contractible in M. Then (M, γ) is an almost product sutured manifold. It is easily observed that (M, γ) is a product sutured manifold if and only if $\iota_* : \pi_1(R_+(\gamma)) \to \pi_1(M)$ is an isomorphism

Proof. Let D ($\subset M$) be a compressing disk for ∂M. Since each component of $s(\gamma)$ is not contractible in M, we may suppose that each component a of $\partial D \cap R(\gamma)$ is an essential arc in $R(\gamma)$. Let (M, γ') be the sutured manifold obtained from (M, γ) by an A-operation along a. Then γ' consists of an annulus which is compressible in ∂M. Hence, by Example 2.7, (M, γ') is an almost product sutured manifold Hence, by Proposition 2.6, (M, γ) is an almost product sutured manifold. □

Example 2.9. Let (V, γ) be a sutured manifold such that V is a genus two handlebody, γ a union of three annuli with $s(\gamma)$ as in Fig. 3 Then (V, γ) is an almost product sutured manifold.

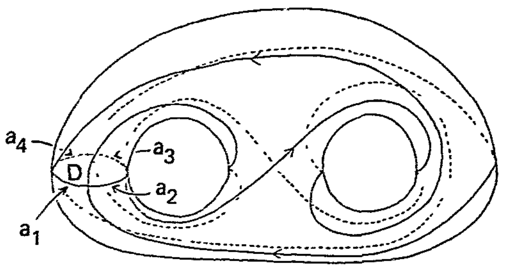


Fig 3

Proof. Let D ($\subset V$) be the properly embedded disk as in Fig 3. Then, $\partial D - s(\gamma)$ consists of four arcs a_1, a_2, a_3, a_4. If (V, γ) is not an almost product sutured manifold, then, by Lemma 2 5, the A-operation along some a_i produces a sutured manifold which is not almost product. Hence we show that A-operations along a_i's produce almost product sutured manifolds. By Lemma 2.5(i), it suffices to show a_1, a_2 produce almost product sutured manifolds. We get the sutured manifold (M_1, γ_1) as in Fig. 4 by the A-operation along a_1. Then do a product decomposition along the disk D' in Fig. 4. Then we get a sutured manifold consisting of a solid torus and a suture which is contractible in the boundary of the solid torus. Hence, by

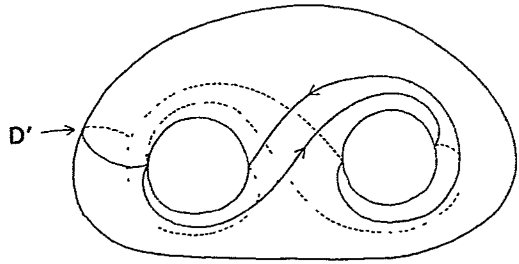


Fig 4

Example 2.7, it is an almost product sutured manifold. In case of a_2, we have the same result. Hence (V, γ) is an almost product sutured manifold □

3. Nonunique minimal genus Seifert surfaces

Let L be an unsplittable link in the 3-sphere, S a minimal genus Seifert surface for L The purpose of this section is to prove

Theorem 3.1. *Suppose that there is a minimal genus Seifert surface S' for L which is not equivalent to S. Then there is a minimal genus Seifert surface S'' for L such that $S'' \cap S = \emptyset$, and S'' is not equivalent to S.*

We note that Scharlemann and Thompson independently obtained a result which is more general [16, Theorem 3.1]

Lemma 3.2 [18, Corollary 3 2]. *Let E be an incompressible surface properly embedded in $F \times I$ such that $\partial E \subset F \times \{0\}$, where F is a compact surface. Then E is parallel to a subsurface of $F \times \{0\}$*

Proof of Theorem 3.1. Firstly, we prepare some notations. We may suppose that $\partial S \cap \partial S' = \emptyset$, and the number of the components of $S \cap S'$ is minimal among all surfaces which are properly isotopic to S'. Let $\tilde{E} \to E(L)$ be the infinite cyclic covering. Then, as described in [14, p. 128], we can get $\tilde{E}$ by taking countably infinite copies of $E(L) - S$, E_i $(i \in \mathbb{Z})$, and identifying their boundaries as in Fig. 5. Then S' lifts homeomorphically to $\tilde{E}$ Let $\tilde{S}'$ $(\subset \tilde{E})$ be a lift of S' Then we assign the integer i to each component of $\tilde{S}' - \bigcup S_j$ which is contained in E_i, where $\{S_j\}$ is the set of all lifts of S to $\tilde{E}$ as in Fig 5. Let s_M (s_m respectively) be the maximal (minimal respectively) value for $\tilde{S}' - \bigcup S_j$. If $s_M = s_m$, then S' is disjoint from S Hence, we suppose that $s_M > s_m$. Let C_M be a component of $S' - S$ corresponding to the components of $\tilde{S}' - \bigcup S_j$ with the the maximal value s_M. Then take sufficiently many parallel copies of S, $S^1, \ldots, S^k$ $(\subset E(L))$ in $N(S)$, and perform the oriented

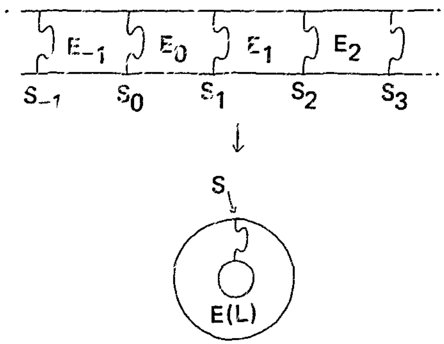


Fig. 5

cut and paste to get an embedded surface $\bar{S}$. By moving S by a tiny isotopy, we may suppose that $\bar{S} \cap S = \emptyset$ (see [15, Lemma 3.2(b)]). Then $\bar{S}$ represents the homology class $(k+1)[S]$ $(\in H_2(E(L), \partial E(L)))$. Hence, by [17, Lemma 1], there is a union of components of $\bar{S}$, S'', such that $S'' \supset C_{M'}$, and $[S''] = [S]$. Since $\chi(\bar{S}) = (k+1)\chi(S)$, we see that $\chi(S'') = \chi(S)$. Then, by pushing S'' to the positive side of S, we may suppose that $S'' \cap (S \cup C_M) = \emptyset$.

Assume that $S \cup S''$ bounds a product region V in $E(L)$, i.e., $(V, S, S', V \cap \partial N(L))$ is homeomorphic to $(S \times I, S \times \{0\}, S \times \{1\}, \partial S \times I)$. By the minimality of $\#(S \cap S')$, we see that each component of $S' - S$ is incompressible in $E(L) - S$. If $C_M \subset V$, then, by Lemma 3.2, we can push C_M to decrease $\#(S \cap S')$, a contradiction. Suppose that C_M is not contained in V. Let C_m be the union of the components of $S' - S$ corresponding to the components of the minimal value s_m. Then, by Lemma 3.2, we see that C_m can be pushed to decrease $\#(S \cap S')$, a contradiction. □

4. Sufficient conditions for the uniqueness

In this section we describe a method for deciding whether the given minimal genus Seifert surface is the unique one of the link. For the methods for finding a minimal genus Seifert surface for the given link, see, for example, [5].

Let L be an unsplittable link in S^3, S a Seifert surface for L. Then the *complementary sutured manifold* [6] for S is the sutured manifold $(M, \gamma) = (\mathrm{cl}(E(L) - N(S)), \mathrm{cl}(\partial E(L) - N(S)))$. Then as an immediate consequence of Theorem 3.1, we have:

Proposition 4.1. *The minimal genus Seifert surfaces for L are unique if and only if there does not exist an incompressible surface S′ in M with* $\partial S' = s(\gamma)$, $\chi(S') = \chi(S)$, *and S′ not parallel to* $R_+(\gamma)$ *and* $R_-(\gamma)$.

Hence we have:

Corollary 4.2. *If (M, γ) is an almost product sutured manifold, then the minimal genus Seifert surfaces for L are unique.*

Then, by Proposition 2 6, we have:

Corollary 4.3. *Let $(M^1, \gamma^1), \ldots, (M^n, \gamma^n)$ be sutured manifolds obtained from the complementary sutured manifold (M, γ) by a sequence of product decompositions and all possible A-operations along the subarcs of the boundary of disks in the ambient manifolds If every (M^i, γ^i) is an almost product sutured manifold, then the minimal genus Seifert surfaces for L are unique*

Example 4.4. Let L be the link which is the boundary of an n-twisted unknotted annulus with $n \neq 0$, i e , L is a $(2, 2n)$-torus link Then the minimal genus Seifert surfaces for L are unique Moreover the annulus is the minimal genus Seifert surface.

Remark. We note that L is fibered if and only if $|n| = 1$.

Proof. It is easy to see that the annulus is a minimal genus Seifert surface Then the complementary sutured manifold is (M, γ) of Example 2 8. Hence, by Corollary 4 2, the minimal genus Seifert surfaces for L are unique □

Example 4.5. Let L be the link as in Fig 6. Then the minimal genus Seifert surfaces for L are unique Moreover the surface S of Fig 6 is the minimal genus Seifert surface.

Proof. It is easy to see the S is a minimal genus Seifert surface It is directly observed that the complementary sutured manifold for S is (V, γ) of Example 2.9. Hence, by Corollary 4.2, the minimal genus Seifert surfaces for L are unique. □

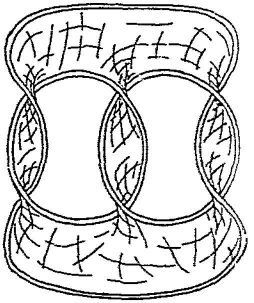


Fig. 6.

In the rest of this section we give a necessary and sufficient condition for the minimal genus Seifert surfaces of the given link being unique in the case when the link has a minimal genus Seifert surface S such that $S^3 - \mathrm{Int}\, N(S)$ is a handlebody

For the statement of the result, we prepare some terminologies Let H be a handlebody (cube with handles). We note that a 3-cell is a handlebody in our context A *complete system of meridian disks* for H is a union of mutually disjoint, properly embedded disks $\mathscr{D}$ in H such that $\mathscr{D}$ cuts H into a 3-cell. Let $\mathscr{H}$ be a union of handlebodies. Then a *complete system of meridian disks* for $\mathscr{H}$ is a union of disks $\mathscr{D}$ in $\mathscr{H}$ such that the restriction of $\mathscr{D}$ to each component of $\mathscr{H}$ is a complete system of meridian disks for the handlebody.

Let (M, γ) be a sutured manifold such that M is a handlebody, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, and (M, γ) satisfies

$$\text{There does not exist an incompressible surface } S \text{ such that } \partial S = s(\gamma), \text{ and } \chi(S) > \chi(R_+(\gamma)). \qquad (*)$$

Then we consider sequences of sutured manifolds of the following type

Firstly, we apply nontrivial product decompositions as much as possible to get a sequence of product decompositions:

$$(M, \gamma) = (M_0, \gamma_0) \to (M_1, \gamma_1) \to \cdots \to (M_k, \gamma_k)$$

We note that each component of M_k is a handlebody. If there is a complete system of meridian disks $\mathscr{D}$ for M_k such that $\partial\mathscr{D} \cap s(\gamma_k) = \emptyset$, then we stop For the method of finding $\mathscr{D}$, see, for example, [10]. If we cannot find such disks, then take a properly embedded disk E in M_k such that $\partial E \cap s(\gamma_k) \neq \emptyset$, and each component of $\partial E \cap R(\gamma_k)$ is an essential arc in $R(\gamma_k)$. Then do A-operations to (M_k, γ_k) along each component of $\partial E - s(\gamma_k)$ to get a number of sutured manifolds: $(M_k, \gamma_{1,k}), \ldots, (M_k, \gamma_{l,k})$

Then do nontrivial product decompositions as much as possible to each $(M_k, \gamma_{i,k})$ to get a sequence of product decompositions

$$(M_k, \gamma_{i,k}) = (M_{i,k}, \gamma_{i,k}) \to (M_{i,k+1}, \gamma_{i,k+1}) \to \cdots \to (M_{i,l_i}, \gamma_{i,l_i}).$$

Then do the same as above to each $(M_{i,l_i}, \gamma_{i,l_i})$, and repeat the process It is not hard to see that we can make the all sequences emanating from (M, γ) as above stop in finitely many steps. Let $(M^1, \gamma^1), \ldots, (M^s, \gamma^s)$ be the sutured manifolds sitting in the ends of the sequences. Let n_i $(1 \leq i \leq s)$ be the length of the sequence from (M, γ) to (M^i, γ^i).

Then we have:

Theorem 4.6. *There exists an incompressible surface S such that $\partial S = s(\gamma)$, and S not parallel to $R_+(\gamma)$ and $R_-(\gamma)$ if and only if there exists (M^i, γ^i) such that $\#(s(\gamma^i)) = \chi(R_+(\gamma)) + n_i$, and (M^i, γ^i) is not an almost product sutured manifold.*

Remark. Since M' is irreducible and each component of $s(\gamma')$ is contractible in m', (M', γ') is almost product if and only if $R_\varepsilon(\gamma')$ ($\varepsilon = +$ or $-$) consists of disks.

Proof of Theorem 4.6. The "only if" part is proved in the proof of Proposition 2 6. Hence we prove the "if" part. For simplicity, we denote the sequence from (M, γ) to (M', γ') by

$$(M, \gamma) = (M_0, \gamma_0) \to (M_1, \gamma_1) \to \cdots \to (M_{n_i}, \gamma_{n_i}) = (M', \gamma').$$

Let S_{n_i} be the union of mutually disjoint, properly embedded disks in M_{n_i} such that $\partial S_{n_i} = s(\gamma_{n_i})$. Then by tracing the constructions of S_2 from S_1 or S' from S in Section 2 conversely, we have surfaces S_j $(0 \leqslant j \leqslant n_i)$ in M_j such that $\partial S_i = s(\gamma_i)$, and

$$\chi(S_j) = \chi(S_{n_i}) - (n_i - j) = \# s(\gamma_{n_i}) - (n_i - j). \tag{$**$}$$

By the assumption $\# s(\gamma_{n_i}) = \chi(R_+(\gamma)) + n_i$, and the above conditions (*), (**), we see that each S_j is incompressible. Hence, by Lemmas 2.2, 2.4, we see that each S_j is not parallel to $R_+(\gamma_j)$ and $R_-(\gamma_j)$, so that $S = S_0$ satisfies the conclusion □

5. The Murasugi sum

The surface R $(\subset S^3)$ is a Murasugi sum of two surfaces R_1 and R_2 in S^3 if:

(1) $R = R_1 \cup_D R_2$, where D is a $2n$-gon, i.e , $\partial D = A_1 \cup B_1 \cup \cdots \cup A_n \cup B_n$ (possibly $n = 1$), where A_i (B_i respectively) is an arc properly embedded in R_1 (R_2 respectively).

(2) there exist 3-balls $\bar{B}_1$, $\bar{B}_2$ in S^3 such that:

(i) $\bar{B}_1 \cup \bar{B}_2 = S^3$, $\bar{B}_1 \cap \bar{B}_2 = \partial \bar{B}_1 = \partial \bar{B}_2 = S$ a 2-sphere,

(ii) $R_1 \subset \bar{B}_1$, $R_2 \subset \bar{B}_2$, and $R_1 \cap S = R_2 \cap S = D$.

Throughout this section, we adopt the above notation. We note that if $n = 1$, then the Murasugi sum is a connected sum. In this section, we prove Theorem 5.1 stated in Section 1.

Proof of the "if" part of Theorem 5.1. Let (M, γ), (M_1, γ_1), (M_2, γ_2) be the complementary sutured manifolds of R, R_1, R_2 respectively. We may suppose that:

(a) $\bar{B}_1 \cap R_-(\gamma) = R_-(\gamma_1)$, $\bar{B}_1 \cap R_+(\gamma) = R_+(\gamma_1) - \operatorname{Int} N(D)$,

(b) $\bar{B}_2 \cap R_-(\gamma) = R_-(\gamma_2) - \operatorname{Int} N(D)$, $\bar{B}_2 \cap R_+(\gamma) = R_+(\gamma_2)$.

Since L_1 is a fibered link, by Gabai [6, Theorem 1.9], there exist mutually disjoint product disks $D_1, \ldots, D_m$ in (M_1, γ_1) such that $\bigcup D_i$ decomposes (M_1, γ_1) into a 3-cell with one suture. By (a), we may suppose that $(D_i \cap R_+(\gamma_1)) \cap \operatorname{Int} N(D) = \emptyset$. Hence, $D_1, \ldots, D_m$ are product disks in (M, γ) and $\bigcup D_i$ decomposes (M, γ) into a sutured manifold which is homeomorphic to (M_2, γ_2). Hence, by Lemma 2 1 and Theorem 3.1, we see that the minimal genus Seifert surfaces for L are unique. □

Proof of the "only if" part of Theorem 5.1. Let $E = S - \operatorname{Int} D$, T the surface obtained by summing R_1 and R_2 along E. By moving T by a tiny isotopy, we may suppose that $R \cap T = \emptyset$ (see Fig. 7) Hence T is properly embedded in (M, γ) and $\partial T = s(\gamma)$. Since the minimal genus Seifert surfaces for L are unique, T is parallel to $R_+(\gamma)$ or $R_-(\gamma)$, say $R_+(\gamma)$ Let (M', γ') be the sutured manifold such that M' is the closure of the component of $M - T$ between T and $R_+(\gamma)$ and $\gamma' = \gamma \cap M'$. Then (M', γ') is a product sutured manifold Let $D'_1, \ldots, D'_n$ be the product disks in (M', γ') which correspond to $A_1 \times I, \ldots, A_n \times I$. Then $\bigcup D'_i$ decomposes (M', γ') into a product sutured manifold $((R_1 - \mathrm{Int}\, N(D)) \times I, \partial(R_1 - \mathrm{Int}\, N(D)) \times I)$ and a sutured manifold homeomorphic to (M_2, γ_2) But since (M', γ') is a product sutured manifold, (M_2, γ_2) is a product sutured manifold Hence L_2 is a fibered link. We note the above argument is essentially the same as in the proof of [6, Corollary 3.2] In that paper, Gabai assumed that the incompressible surfaces for L are unique. But what he used in the proof was the uniqueness of the minimal genus Seifert surfaces for L

Now we will show that the minimal genus Seifert surfaces for L_1 are unique Assume that the minimal genus Seifert surfaces for L_1 are not unique Then, by Theorem 3 1, there is a minimal genus Seifert surface T_1 of L_1 such that $R_1 \cap T_1 = \emptyset$ and T_1 is not isotopic to R_1 Since $R_1 \cap T_1 = \emptyset$, we may suppose that $T_1 \cap \bar{B}_2 = \emptyset$, and, moreover, T_1 looks like as in Fig 8 in the neighborhood of D. We get R from

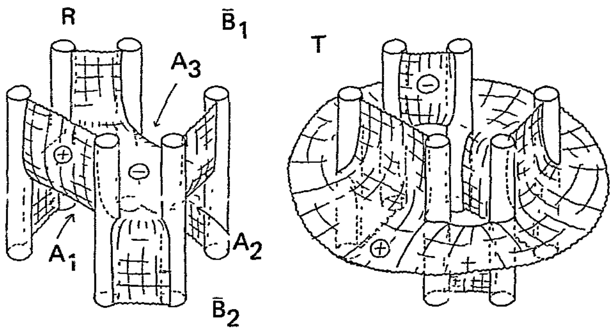


Fig 7

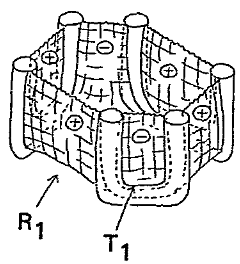


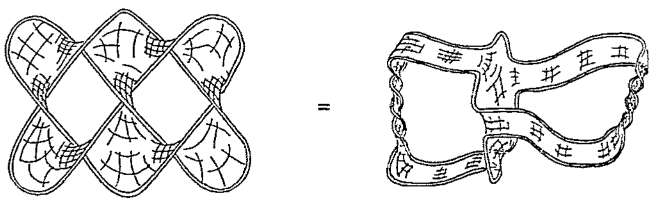
Fig 8

R_1 by attaching $R_2 - \text{Int}\, D$ along the arcs $B_1, \ldots, B_n$ Similarly, we can get a minimal genus Seifert surface R' for L from T_1 by attaching $R_2 - \text{Int}\, D$ by using the arcs $B_1, \ldots, B_n$. Then, by moving R' by a tiny isotopy, we may suppose that $R' \cap R = \emptyset$, $R' \cap T = \emptyset$. Suppose that R' is parallel to $R_-(\gamma)$ in M. Let (N, δ) be the product sutured manifold between R and R', and $D_1'', \ldots, D_n''$ the product disks in (N, δ) corresponding to $B_1 \times I, \ldots, B_n \times I$ By decomposing (N, δ) along $\bigcup D_i'$ we get a product sutured manifold $(\text{cl}(R_2 - D) \times I, \partial(R_2 - D) \times I)$ and a sutured manifold homeomorphic to the closure of the region between T_1 and $R_-(\gamma_1)$ in M_1 Hence we see that T_1 is parallel to $R_-(\gamma_1)$, a contradiction. Suppose that R' is parallel to $R_+(\gamma)$ in M. Then R' is parallel to T in M Then, by applying the same arguments, we see that R' is not parallel to T, a contradiction. Hence the minimal genus Seifert surfaces for L are not unique, a contradiction $\square$

Then Corollary 5.2 in Section 1 is an immediate consequence of Example 4.4, and Theorem 5 1.

Example 5.3. The minimal genus Seifert surfaces for 7_4 (see [14, Appendix C]) are not unique.

Proof. Let S be a minimal genus Seifert surface for 7_4 as in Fig 9. By Fig. 9(ii), we see that S is a Murasugi sum of two annuli bounding $(2, 4)$-torus links. We note that a $(2, 4)$-torus link is not fibered Hence, by Theorem 5.1, the minimal genus Seifert surfaces for 7_4 are not unique $\square$



(i) (ii)

Fig 9

Example 5.4. The minimal genus Seifert surfaces for 9_{25} are unique.

Proof. Let S be a minimal genus Seifert surface for 9_{25} as in Fig 10. By Fig. 10(ii), we see that S can be represented as a Murasugi sum of two annuli bounding $(2, 2)$-torus links and a genus-0 surface bounding $(2, 2, 2)$-pretzel link. We note that a $(2, 2)$-torus link is fibered. Hence, by Example 4.5, and Theorem 5.1, we see that the minimal genus Seifert surfaces for 9_{25} are unique.

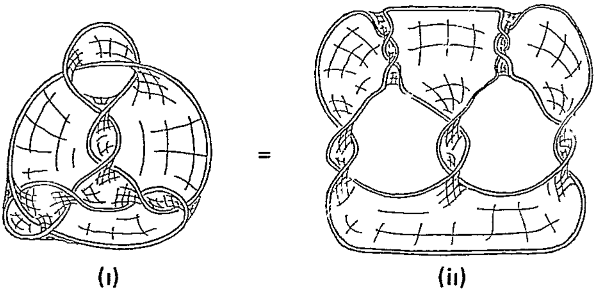


Fig 10

Appendix

The list of knots of ≤ 10 crossings whose minimal genus Seifert surfaces are not unique For the notations, see [14, Appendix C]

7_4, 8_3, 9_5, 9_{10}, 9_{13}, 9_{18}, 9_{23}, 10_3, 10_{11}, 10_{16}, 10_{18}, 10_{24}, 10_{28}, 10_{30}, 10_{31}, 10_{33}, 10_{37}, 10_{38}, 10_{53}, 10_{67}, 10_{68}, 10_{74}.

Note added in proof

Dr. Osamu Kakimizu pointed out that the "only if" part of Proposition 4 1 holds only in case when S is connected. However this does not affect other results of this paper because we did not use the "only if" part

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